J Glob Optim (2008) 42:157–175
DOI 10.1007/s10898-007-9236-z

On *Saaty*'s and *Koczkodaj*'s inconsistencies of pairwise comparison matrices

Sándor Bozóki · Tamás Rapcsák

Received: 11 September 2007 / Accepted: 13 September 2007 / Published online: 13 October 2007

Abstract The aim of the paper is to obtain some theoretical and numerical properties of *Saaty*'s and *Koczkodaj*'s inconsistencies of pairwise comparison matrices (*PRM*). In the case of 3×3 *PRM*, a differentiable one-to-one correspondence is given between *Saaty*'s inconsistency ratio and *Koczkodaj*'s inconsistency index based on the elements of *PRM*. In order to make a comparison of *Saaty*'s and *Koczkodaj*'s inconsistencies for 4×4 pairwise comparison matrices, the average value of the maximal eigenvalues of randomly generated $n \times n$ *PRM* is formulated, the elements a_{ij} $(i < j)$ of which were randomly chosen from the ratio scale

$$\frac{1}{M}, \frac{1}{M-1}, \ldots, \frac{1}{2}, 1, 2, \ldots, M-1, M,$$

with equal probability $1/(2M-1)$ and a_{ji} is defined as $1/a_{ij}$. By statistical analysis, the empirical distributions of the maximal eigenvalues of the *PRM* depending on the dimension number are obtained. As the dimension number increases, the shape of distributions gets similar to that of the normal ones. Finally, the inconsistency of asymmetry is dealt with, showing a different type of inconsistency.

Keywords Pairwise comparison matrix · Inconsistency · Inconsistency index · Randomly generated pairwise comparison matrix

1 Introduction

In multiattribute decision making (*MADM*), the aim is to rank a finite number of alternatives with respect to a finite number of attributes. Tender evaluations, public procurement pro-

S. Bozóki · T. Rapcsák (✉)
Computer and Automation Research Institute, Hungarian Academy of Sciences, Budapest, Hungary
e-mail: rapcsak@oplab.sztaki.hu

S. Bozóki
e-mail: bozoki@oplab.sztaki.hu

cesses, selections of applicants for positions, decisions on the best portfolios in investments are real-life decision situations in which *MADM* models can be used.

In solving a multiattribute decision problem, one needs to know the importances or weights of the not equally important attributes and also the evaluations of the alternatives with respect to the attributes. One technique, often used, is the method of pairwise comparisons a concept which is more than 200 years old. Condorcet (1785) and Borda (1781) introduced it for voting problems in the 1780s by using only 0 and 1 in the pairwise comparison matrices. In experimental psychology, Thorndike (1920) and Thurstone (1927) used it in the 1920s. Especially, pairwise comparisons based on a ratio scale is one of the basic pillars of the *Analytic Hierarchy Process* (Saaty 1980).

Given n objects, e.g., attributes or alternatives, we suppose that the decision maker(s) is (are) able to compare any two of them. In preference modelling, this assumption is called comparability. For any pairs $(i, j),\ i, j = 1, 2, \ldots, n$, the decision maker is requested to tell how many times the ith object is preferred (or more important) than the jth one, which result is denoted by a_{ij}.

By definition,

$$a_{ij} > 0; \tag{1.1}$$

$$a_{ii} = 1; \tag{1.2}$$

$$a_{ij} = \frac{1}{a_{ji}}, \tag{1.3}$$

for any pair of indices $(i, j),\ i, j = 1, 2, \ldots, n$. The name of matrices $\mathbf{A} = [a_{ij}]_{i,j=1,2,\ldots,n} \in R^{n\times n}$ with properties (1.1–1.3) is pairwise comparison matrices or positive reciprocal matrices (*PRM*).

A pairwise comparison matrix $\mathbf{A}$ is consistent if it satisfies the transitivity property

$$a_{ij}a_{jk} = a_{ik} \tag{1.4}$$

for any indices $(i, j, k),\ i, j, k = 1, 2, \ldots, n$. Otherwise, $\mathbf{A}$ is inconsistent. It was shown by Saaty (1980) that a pairwise comparison matrix is consistent if and only if it is of rank one. When a pairwise comparison matrix $\mathbf{A}$ is consistent, the normalized weights computed from $\mathbf{A}$ are unique. Otherwise, an approximation of $\mathbf{A}$ by a consistent matrix (determined by a vector) is needed.

A crucial point of this methodology is to determine the inconsistency of the pairwise comparison matrices. The only widely accepted rule of inconsistency is due to Saaty (1980), but his definition does not meet some important requirements (see Sect. 2). The aim of the paper is to make some comparison on *Saaty*'s and *Koczkodaj*'s inconsistencies of pairwise comparison matrices. The two approaches seem to be completely different, because while *Saaty*'s inconsistency ratio is an index for the departure from randomness, *Koczkodaj*'s inconsistency index is related to the departure from consistency with the possibility to locate inconsistency.

In Sect. 2, the question is how to investigate *Saaty*'s and *Koczkodaj*'s inconsistencies. In Sect. 3, the inconsistency formulas of 3×3 pairwise comparison matrices are studied from theoretical and computational points of view. A differentiable one-to-one correspondence is given between *Saaty*'s and *Koczkodaj*'s inconsistencies. In Sect. 4, by using statistical tools, the average value of the maximal eigenvalues of randomly generated $n \times n$ *PRM* is formulated, the elements a_{ij} $(i < j)$ of which were randomly chosen from the ratio scale $\frac{1}{M}, \frac{1}{M-1}, \ldots, \frac{1}{2}, 1, 2, \ldots, M-1, M$, with equal probability $1/(2M-1)$ and a_{ji} is defined as $1/a_{ij}$. Then, a comparison of *Koczkodaj*'s inconsistency index and *Saaty*'s inconsistency ratio is given for 4×4 pairwise comparison matrices. In Sect. 5, the inconsistency of random pairwise comparison matrices is investigated and by statistical analysis, the empirical distributions of the maximal eigenvalues of the *PRM* depending on the dimension number are obtained. As the dimension number increases, the shape of distributions gets similar to that of the normal ones. In Sect. 6, the inconsistency of asymmetry is dealt with, showing a different type of inconsistency.

2 Inconsistency indices

In real-life decision problems, pairwise comparison matrices are rarely consistent. Nevertheless, decision makers are interested in the level of consistency of the judgements, which somehow expresses the goodness or "harmony" of pairwise comparisons totally, because inconsistent judgements may lead to senseless decisions.

Saaty (1980) proposed the following method for calculating inconsistency. Computing the largest eigenvalue $\lambda_{\max}$ of $\mathbf{A}$, he has shown that $\lambda_{\max} \geq n$ and equals to n if and only if $\mathbf{A}$ is consistent. Then, inconsistency index (CI_n) is defined by

$$CI_n = \frac{\lambda_{\max} - n}{n - 1},$$

which gives the average inconsistency. Mathematically, inconsistency is not but a rescaling of the largest eigenvalue. Since $\lambda_{\max} \geq n$, CI_n is always non-negative. The inconsistency index in its own has no meaning, unless we compare it with some benchmark to determine the magnitude of the deviation from consistency. Let a set of e.g., 500 random pairwise comparison matrices of size $n \times n$ be generated so that each element a_{ij} $(i < j)$ be randomly chosen from the scale

$$\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, \frac{1}{2}, 1, 2, \ldots, 8, 9,$$

and a_{ji} is defined as $\frac{1}{a_{ij}}$. Let RI_n denote the average value of the randomly obtained inconsistency indices, which depends not only on n but on the method of generating random numbers, too. The inconsistency ratio (CR_n) of a given pairwise comparison matrix $\mathbf{A}$ indicating inconsistency is defined by

$$CR_n = \frac{CI_n}{RI_n}.$$

If the matrix is consistent, then $\lambda_{\max} = n$, so $CI_n = 0$ and $CR_n = 0$, as well. *Saaty* concluded that an inconsistency ratio of about 10% or less may be considered acceptable. The intuitive meaning of the 10% rule is skipped by several authors. A statistical interpretation of the 10% rule is given by Vargas (1982). More recently, *Saaty*'s threshold is 5% for 3×3, and 8% for 4×4 matrices (Saaty 1994).

It is emphasized that the inconsistency ratio CR_n is related to *Saaty*'s scale. The structuring process in AHP specifies that items to be compared should be within one order of magnitude. This helps avoid inaccuracy associated with cognitive overload as well as $a_{ij}a_{jk}$ relationships that are beyond the 1–9 scale, see e.g. Lane and Verdini (1989) and Murphy (1993). If only two attributes (or alternatives) are present, inconsistency is always zero, since the decision maker gives only one importance ratio.

Though the only one widely accepted rule of inconsistency for any order of matrix is due to Saaty, its consistency definition has some drawbacks. By Koczkodaj (1993), "The author of this paper truly believes that failure of the pairwise comparison method to become more popular has its roots in the consistency definition." The major drawback of *Saaty*'s inconsistency definition seems to be the 10% rule of thumb. Another weakness of it is related to the location of inconsistency or rather its lack. Since an eigenvalue is a global characteristic of a matrix, by examining it, we cannot say which matrix element contributed to the increase of inconsistency. Some improvements can be found in Saaty (1990).

A general 3×3 pairwise comparison matrix has three comparisons a, b, c. In order to define *Koczkodaj*'s inconsistency index (Duszak and Koczkodaj 1994; Koczkodaj 1993), consider a general 3×3 pairwise comparison matrix. Reduce this reciprocal matrix to a vector of three coordinates (a, b, c). In the consistent cases, the equality $b = ac$ holds. It is always possible to produce three consistent reciprocal matrices (represented by three vectors) by computing one coordinate from the combination of the remaining two coordinates. These three vectors are: $\left(\frac{b}{c}, b, c\right)$, (a, ac, c) and $\left(a, b, \frac{b}{a}\right)$.

The inconsistency index of a general 3×3 pairwise comparison matrix is defined by Koczkodaj as the relative distance to the nearest consistent 3×3 pairwise comparison matrix represented by one of these three vectors.

Definition 2.1 The inconsistency index of a general 3×3 pairwise comparison matrix is equal to

$$CM(a, b, c) = \min \left\{ \frac{1}{a}\left|a - \frac{b}{c}\right|, \ \frac{1}{b}\left|b - ac\right|, \ \frac{1}{c}\left|c - \frac{b}{a}\right| \right\}. \tag{2.1}$$

The inconsistency index of an $n \times n$ $(n > 2)$ reciprocal matrix A is equal to

$$CM(A) = \max \left\{ \min \left\{ \left|1 - \frac{b}{ac}\right|, \left|1 - \frac{ac}{b}\right| \right\} \quad \text{for each triad } (a, b, c) \text{ in } A \right\}. \tag{2.2}$$

In the case of matrices of higher orders, the inconsistency index of a matrix element is equal to the maximum of CM of all possible triads which include this element.

Note that the inconsistency index is not a metric. By Duszak and Koczkodaj (1994), the number of all possible triads of the $n \times n$ comparison matrices is equal to

$$n(n-1)(n-2)/3!. \tag{2.3}$$

In the case of 4×4 pairwise comparison matrices and a scale of 1–5, the threshold should be 1/3 (Koczkodaj et al. 1997).

Other inconsistency indices have been introduced. The inverse inconsistency index suggested by Dodd et al. (1993), Monsuur (1996) applied a transformation of the maximal eigenvalues, Peláez and Lamata (2003) examined all the triples of elements and used the determinant to indicate consistency, furthermore, Stein and Mizzi (2007) obtained the harmonic consistency index. Another type of inconsistency index is the distance from a specific consistent matrix. Chu et al. (1979) used the least squares estimation error, Crawford and Williams (1985) the logarithmic least squares estimation error, furthermore, Aguarón and Moreno-Jiménez (2003) the geometric consistency index for the logarithmic least squares method (the row geometric mean method).

Table 1 summarizes some weighting methods and inconsistency indices, namely, the eigenvector method (*EM*) and inconsistency ratio (*CR*) (Saaty 1980), the least squares method (*LSM*) (Chu et al. 1979), the chi squares method ($\chi^2 M$) (Jensen 1983), the singular value decomposition method (*SVDM*) (Gass and Rapcsák 2004) and *Koczkodaj*'s inconsistency index (Koczkodaj 1993), the logarithmic least squares method (*LLSM*) (Crawford and Williams 1985) and *GCI* (Aguarón and Moreno-Jiménez 2003).

Table 1 Weighting methods and inconsistency indices

Method	The problem to be solved	Inconsistency (The optimal solution is denoted by **w**)	Threshold of acceptability
Eigenvector method, EM	$\lambda_{\max}\mathbf{w} = \mathbf{A}\mathbf{w}$, $\sum_{i=1}^{n} w_i = 1$	$CR_n = \frac{\frac{\lambda_{\max}-n}{n-1}}{RI_n}$, where RI_n denotes the average CI value of $n \times n$ random matrices	$CR_n \leq 0.1$
Least squares method, LSM	$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{w_i}{w_j}\right)^2$ $\sum_{i=1}^{n} w_i = 1, w_i > 0,\ i = 1, 2, \ldots, n$	$\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{w_i^{LSM}}{w_j^{LSM}}\right)^2}$	
Chi squares method, $\chi^2 M$	$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(a_{ij}-\frac{w_i}{w_j}\right)^2}{\frac{w_i}{w_j}}$ $\sum_{i=1}^{n} w_i = 1, w_i > 0, \quad i = 1, 2, \ldots, n$	$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(a_{ij}-\frac{w_i^{\chi^2 M}}{w_j^{\chi^2 M}}\right)^2}{\frac{w_i^{\chi^2 M}}{w_j^{\chi^2 M}}}$	$CM(\mathbf{A}) \leq 0.33$ $n = 4$ scale of $1, \ldots, 5$
Singular value decomposition method, $SVDM$	$\mathbf{A}_{[1]} = \alpha_1 \mathbf{u}\mathbf{v}^T$ the best one rank approximation of $\mathbf{A}$ in Frobenius norm; $w_i^{SVD} = \frac{u_i+\frac{1}{v_i}}{\sum_{j=1}^{n}\left(u_j+\frac{1}{v_j}\right)} \quad i = 1, 2, \ldots, n$	$\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{w_i^{SVD}}{w_j^{SVD}}\right)^2}$	
Logarithmic least squares method, LLSM	$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\ln a_{ij} - \ln \frac{w_i}{w_j}\right)^2$ $\sum_{i=1}^{n} w_i = 1,$ $w_i > 0,\ i = 1, 2, \ldots, n$	$GCI(\mathbf{A}) = \frac{2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\ln a_{ij} - \ln \frac{w_i^{LLSM}}{w_j^{LLSM}}\right)^2}{(n-1)(n-2)}$	$GCI(\mathbf{A}) \leq 0.3147$ $n = 3$ $GCI(\mathbf{A}) \leq 0.3526$ $n = 4$ $GCI(\mathbf{A}) \leq 0.370$ $n > 4$

3 Inconsistency of 3 × 3 pairwise comparison matrices

In this part, it is shown that there exists a one-to-one correspondence between *Saaty*'s inconsistency ratio and *Koczkodaj*'s inconsistency index.

The general form of 3×3 positive reciprocal matrices is as follows:

$$\begin{pmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{pmatrix}, \quad a, b, c \in R_+. \tag{3.1}$$

By Tummala and Ling (1998), the maximal eigenvalues of matrices (3.1) can be explicitly given by the function

$$\lambda_{\max}(a, b, c) = 1 + \sqrt[3]{\frac{b}{ac}} + \sqrt[3]{\frac{ac}{b}}, \quad (a, b, c) \in R_+^3. \tag{3.2}$$

A consequence of this formula is that $\lambda_{\max}$ does not change if the elements a and b are multiplied by the same constant. Thus, the *CR*-inconsistencies of matrices

$$\begin{pmatrix} 1 & 2 & 2 \\ & 1 & 2 \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 7 & 7 \\ & 1 & 2 \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 9 & 9 \\ & 1 & 2 \\ & & 1 \end{pmatrix} \tag{3.3}$$

are equal, though the consistency violations in the matrices are different.

By formula (3.2), it is possible to make a connection between $\lambda_{\max}$ and the inconsistency originated from the elements a, b, c of the positive reciprocal matrices.

Definition 3.1 In the case of (3.1), let T denote the maximum of two ratios, $\frac{ac}{b}$ and $\frac{b}{ac}$, i.e., $T = \max\left\{\frac{ac}{b}, \frac{b}{ac}\right\}$.

If the matrix is consistent, T equals to 1, otherwise, $T > 1$.

Theorem 3.1 *In the case of* 3×3 *pairwise comparison matrices, there exists a differentiable one-to-one correspondence for every pair of the inconsistency CR defined by Saaty, the inconsistency CM defined by Koczkodaj and* $T = \max\left\{\frac{ac}{b}, \frac{b}{ac}\right\}$ *as follows:*

$$CR(T) = \frac{\sqrt[3]{T} + \frac{1}{\sqrt[3]{T}} - 2}{2RI_3}, \quad T > 1. \tag{3.4}$$

$$T(CR) = \left(1 + RI_3\, CR + \sqrt{RI_3\, CR(2 + RI_3\, CR)}\right)^3, \quad CR \in (0, \infty), \tag{3.5}$$

$$CM(T) = 1 - \frac{1}{T}, \quad T(CM) = \frac{1}{1 - CM}, \quad CM \in (0, 1), \tag{3.6}$$

$$CR(CM) = \frac{\frac{1}{\sqrt[3]{1 - CM}} + \sqrt[3]{1 - CM} - 2}{2RI_3}, \quad CM \in (0, 1), \tag{3.7}$$

$$CM(CR) = 1 - \frac{1}{\left(1 + RI_3 CR + \sqrt{RI_3 CR(2 + RI_3 CR)}\right)^3}, \quad CR \in (0, \infty). \tag{3.8}$$

Proof From Definition 3.1, it follows that

$$\sqrt[3]{\frac{ac}{b}} + \sqrt[3]{\frac{b}{ac}} = \sqrt[3]{T} + \frac{1}{\sqrt[3]{T}}.$$

Since $\lambda_{\max} = 1 + \sqrt[3]{\frac{ac}{b}} + \sqrt[3]{\frac{b}{ac}}$, it can be written in the equivalent form

$$\lambda_{\max} = 1 + \sqrt[3]{T} + \frac{1}{\sqrt[3]{T}}. \tag{3.9}$$

Saaty defined the inconsistency ratio as $CR = \frac{\frac{\lambda_{\max}-n}{n-1}}{RI_n}$. Let us substitute $n = 3$ and (3.9) for the formula of CR, and (3.4) is proved.

Function $CR(T)$ is differentiable on the domain $T > 1$, and

$$CR'(T) = \frac{1 - \frac{1}{\sqrt[3]{T}^2}}{6RI_3\sqrt[3]{T}^2}, \tag{3.10}$$

which is positive if $T > 1$, consequently, CR is invertable in this domain. Its inverse function is equal to

$$T(CR) = \left(1 + RI_3\,CR + \sqrt{RI_3\,CR(2 + RI_3\,CR)}\right)^3, \quad CR \in (0, \infty),$$

which proves (3.5).

Since

$$\begin{aligned} CM &= \min\left\{\frac{1}{a}\left|a - \frac{b}{c}\right|, \frac{1}{b}\left|b - ac\right|, \frac{1}{c}\left|c - \frac{b}{a}\right|\right\} \\ &= \min\left\{\left|1 - \frac{b}{ac}\right|, \left|1 - \frac{ac}{b}\right|, \left|1 - \frac{b}{ac}\right|\right\} = \min\left\{\left|1 - \frac{b}{ac}\right|, \left|1 - \frac{ac}{b}\right|\right\}, \end{aligned}$$

it follows that

$$CM(T) = 1 - \frac{1}{T}, \quad CM'(T) = \frac{1}{T^2}, \quad T > 1,$$

and

$$T(CM) = \frac{1}{1 - CM}, \quad T'(CM) = \frac{1}{(1 - CM)^2}, \quad CM \in (0, 1).$$

In order to obtain $CR(CM)$, formulas (3.4) and (3.6) are used:

$$CR(CM) = \frac{\frac{1}{\sqrt[3]{1 - CM}} + \sqrt[3]{1 - CM} - 2}{2RI_3}, \quad CM \in (0, 1). \tag{3.11}$$

Similarly, formulas (3.5) and (3.6) are used to obtain

$$CM(CR) = 1 - \frac{1}{\left(1 + RI_3CR + \sqrt{RI_3CR(2 + RI_3CR)}\right)^3}, \quad CR \in (0, \infty). \tag{3.12}$$

Since the derivatives

$$\begin{aligned} CR'(CM) &= CR'(T)\,T'(CM) \quad \text{and} \\ CM'(CR) &= CM'(T)\,T'(CR) \end{aligned}$$

are different from zero, we have one-to-one correspondences. □

Corollary 3.1 *In the case of* 3×3 *pairwise comparison matrices, the following properties are equivalent:*

$$CR \leq 10\% \tag{3.13}$$

$$\frac{1}{2.63} = 0.38 \leq \frac{ac}{b} \leq 2.63; \tag{3.14}$$

$$CM \leq 0.62. \tag{3.15}$$

Proof (3.13) $\Leftrightarrow$ (3.14): Let $x = \sqrt[3]{\frac{b}{ac}}$. From (3.2) and since $\lambda_{\max}$ corresponding to $CR = 10\%$ is 3.1048, (3.13) is equivalent to

$$x^2 - 2.1x + 1 \leq 0, \quad x > 0.$$

By solving equality $x^2 - 2.1x + 1 = 0,\ x > 0$, we obtain that $x_1^* \approx 1.38$ and $x_2^* = \frac{1}{x_1^*} \approx 0.7244$. Thus,

$$\frac{1}{x^*} \leq \sqrt[3]{\frac{b}{ac}} \leq x_1^*,$$

which is equivalent to the statement.

(3.13) $\Leftrightarrow$ (3.15) follows from (3.11) and (3.12). $\square$

The intuitional meaning of (3.13) $\Leftrightarrow$ (3.14) in Theorem 3.1 may be interpreted by the following example. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{pmatrix}.$$

Now, $a = 2,\ b = 6,\ c = 3$, and $\mathbf{A}$ is consistent $\left(\frac{ac}{b} = 1\right)$. Let us fix a and b. If, e.g., $c = 4$, the inconsistency of matrix $\mathbf{A}$ remains acceptable, because

$$\frac{ac}{b} = \frac{2 \cdot 4}{6} = 1.33 < 2.63.$$

The maximal value of c, for which matrix $\mathbf{A}$ is acceptable by the 10% rule, is $3 \times 2.63 = 7.89$.

We remark that the CM-inconsistencies of matrices (3.3) are equal as well.

4 A comparison of *Saaty*'s and *Koczkodaj*'s inconsistency indices for 4 × 4 pairwise comparison matrices

Koczkodaj (1997) reported on concrete inconsistency index calculations based on a ratio scale $1/5, 1/4, 1/3, 1/2, 1, 2, 3, 4, 5$ for 4×4 pairwise comparison matrices. He remarked that in this case, an acceptable threshold of inconsistency is 1/3. In order to make comparisons between *Saaty*'s and *Koczkodaj*'s inconsistency indices, we have to fit *Saaty*'s threshold to the ratio scale $1/5, 1/4, 1/3, 1/2, 1, 2, 3, 4, 5$.

By the definition of *CR*, the rule of acceptability of a pairwise comparison matrix is that the maximal eigenvalue $\lambda_{\max}$ should not be greater than a linear combination of the average

$\lambda_{\max}$ of randomly generated matrices, denoted by $\overline{\lambda_{\max}}$, with a coefficient 0.1, and $\lambda_{\max} (= n)$ of a consistent matrix, with a coefficient 0.9, i.e.,

$$CR \leq 0.10 \Longleftrightarrow \lambda_{\max} \leq 0.1\overline{\lambda_{\max}} + 0.9n. \tag{4.1}$$

We remark that $\overline{\lambda_{\max}}$ grows more rapidly (the slope of the approximating line is 2.76) than n.

Let $\bar{\lambda}_{\max}(n, M)$ denote the average value of the dominant eigenvalue of a randomly generated $n \times n$ matrix the elements of which are chosen from the ratio scale

$$\frac{1}{M}, \frac{1}{M-1}, \ldots, \frac{1}{2}, 1, 2, \ldots, M-1, M, \tag{4.2}$$

with equal probability $\frac{1}{2M-1}$.
Table 2 presents the values of $\bar{\lambda}_{\max}(n, M)$ for $n = 3, 4, \ldots, 10$ and $M = 3, 4, \ldots, 15$. $\bar{\lambda}_{\max}(n, M)$ can be well approximated by using a 4-parameter quasilinear regression.

Theorem 4.1

$$\bar{\lambda}_{\max} = 0.5625n - 0.621M + 0.2481Mn + 1.1478 + \varepsilon(n, M), \tag{4.3}$$

where $\varepsilon(n, M)$ denotes the approximation error of $\bar{\lambda}_{\max}(n, M)$.

Proof The least-squares optimal solution of the 4-parameter quasilinear approximation problem

$$\bar{\lambda}_{\max}(n, M) \approx \alpha n + \beta M + \gamma n M + \delta$$

is as follows:

$$\begin{aligned} \alpha &= 0.5625, \\ \beta &= -0.6210, \\ \gamma &= 0.2481, \\ \delta &= 1.1478. \end{aligned}$$

The maximal approximate error $\varepsilon(n, M)$, while $3 \leq n \leq 10$, $3 \leq M \leq 15$, is 0.35. □

Let $CI(n, M)$, $RI(n, M)$ and $CR(n, M)$ denote the inconsistency index, the average value of the randomly obtained inconsistency indices and the inconsistency ratio with respect to the dimension number n and ratio scale (4.2), respectively. The theorem above provides an equivalent characterization of the 10% rule as follows:

Corollary 4.1

$$\begin{aligned} CR(n, M) = \frac{CI(n, M)}{RI(n, M)} \leq 0.10 \Longleftrightarrow \quad & \lambda_{\max} \leq 0.95625n - 0.0621M \\ & +0.02481Mn + 0.1148. \end{aligned} \tag{4.4}$$

Proof By substituting (4.3) for (4.1), we have the result. □

We emphasize that the condition for the acceptable inconsistency in (4.4) depends only on the data of the experimental pairwise comparison matrix, namely, on its dimension and

Table 2 Average value of the largest eigenvalues of random PRM depending on the largest element of the ratio scale

$\lambda_{max}(n,M)$		The largest element (M)												
		3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Matrix size (n)	**3**	3.236	3.369	3.505	3.641	3.778	3.913	4.049	4.184	4.317	4.450	4.582	4.714	4.845
	4	4.555	4.884	5.226	5.578	5.933	6.292	6.652	7.015	7.378	7.742	8.106	8.472	8.834
	5	5.901	6.445	7.017	7.606	8.208	8.819	9.435	10.057	10.683	11.311	11.940	12.574	13.209
	6	7.256	8.019	8.823	9.656	10.506	11.370	12.245	13.128	14.017	14.913	15.811	16.712	17.620
	7	8.615	9.597	10.633	11.705	12.801	13.915	15.045	16.185	17.331	18.486	19.645	20.810	21.978
	8	9.977	11.177	12.442	13.752	15.091	16.452	17.830	19.220	20.620	22.028	23.445	24.865	26.290
	9	11.339	12.757	14.252	15.797	17.377	18.983	20.605	22.244	23.895	25.555	27.222	28.896	30.574
	10	12.702	14.337	16.059	17.840	19.658	21.504	23.373	25.258	27.155	29.063	30.980	32.903	34.832

Table 3 Average value of the inconsistency indices of random PRM depending on the largest element of the ratio scale

$RI(n, M)$		The largest element (M)												
		3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Matrix size (n)	**3**	0.118	0.185	0.252	0.321	0.389	0.457	0.525	0.592	0.658	0.725	0.791	0.857	0.922
	4	0.185	0.295	0.409	0.526	0.644	0.764	0.884	1.005	1.126	1.247	1.369	1.491	1.611
	5	0.225	0.361	0.504	0.651	0.802	0.955	1.109	1.264	1.421	1.578	1.735	1.894	2.052
	6	0.251	0.404	0.565	0.731	0.901	1.074	1.249	1.426	1.603	1.783	1.962	2.142	2.324
	7	0.269	0.433	0.606	0.784	0.967	1.153	1.341	1.531	1.722	1.914	2.108	2.302	2.496
	8	0.282	0.454	0.635	0.822	1.013	1.207	1.404	1.603	1.803	2.004	2.206	2.409	2.613
	9	0.292	0.470	0.657	0.850	1.047	1.248	1.451	1.656	1.862	2.069	2.278	2.487	2.697
	10	0.300	0.482	0.673	0.871	1.073	1.278	1.486	1.695	1.906	2.118	2.331	2.545	2.759

its largest element. If we use a continuous ratio scale instead of the discrete scale by Saaty, the results remain almost the same.

The results of Theorem 4.1 and Corollary 4.1 can be used in the case of experimental pairwise comparison matrices. A set of 384 *PRM* taken from real-world AHP analyses were studied in Gass and Standard (2002). The experimental distribution of the numbers in the basic AHP comparison scale was unexpected. It seems that for these real-world problems, the decision makers did not use with large experimental probability the extreme comparison values of 8 and 9 (see Table 1 in Gass and Standard (2002). Consequently, in order to estimate the inconsistency more precisely, the influence of the pairwise comparisons determined by the decision makers can be taken into consideration through the largest ratio numbers, respectively.

Based on Theorem 4.1 and Table 3, the inconsistency ratio $CR(4,5)$ can be determined. By generating all the possible *PRM* ($9^6 = 531,441$ matrices) with $CM \leq 1/3$ (1,377 matrices) on the ratio scale $1/5, \ldots, 1, \ldots, 5$, Fig. 1 shows that the possible values of CM under 1/3 are from the set $\{0, 1/6, 1/5, 1/4, 1/3\}$ and the total number of different pairs $(CM, CR(4,5))$ is 14. We can state that the threshold $CM \leq 1/3$ corresponds to $CR(4,5) \leq 0.0336(3.36\%)$. It follows that *Koczkodaj*'s inconsistency index for 4×4 pairwise comparison matrices with respect to ratio scale $1/5, \ldots, 1, \ldots, 5$, is stricter than that of *Saaty*'s. It is noted that the 10% rule allows much higher CM-inconsistency when using the ratio scale $1/9, \ldots, 1, \ldots, 9$. An example is as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 1/8 & 2 & 6 \\ 8 & 1 & 7 & 9 \\ 1/2 & 1/7 & 1 & 2 \\ 1/6 & 1/9 & 1/2 & 1 \end{pmatrix},$$

where $CR = CR(4,9) = 9.47\%$ and $CM = 0.8125$.

It is emphasized that the threshold $CM \leq 1/3$ is given for 4×4 pairwise comparison matrices with respect to the ratio scale $1/5, \ldots, 1, \ldots, 5$. A question arises, namely, how to determine the threshold values for higher dimensions. A possible way is to use the "one grade off" or "two grades off" rules. By Koczkodaj (1997), "An acceptable threshold of inconsistency is 0.33 because it means that one judgement is not more than two grades of the scale "different" from the remaining two judgements."

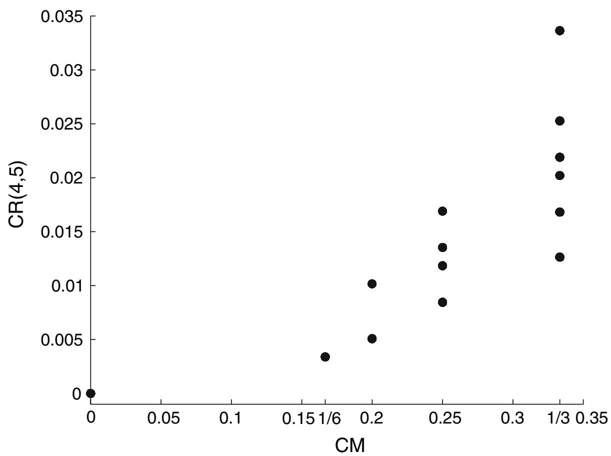
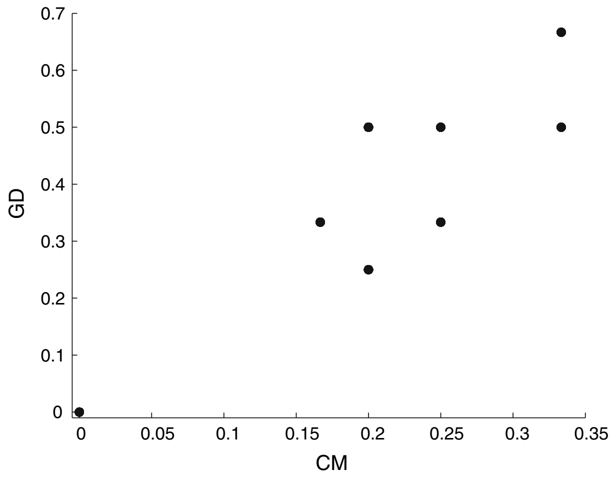


Fig. 1 *Koczkodaj*'s $CM \leq 1/3$ rule corresponds to $CR(4,5) \leq 3.36\%$

Fig. 2 $CM \leq 1/3$ threshold corresponds to $GD \leq 2/3$



Let us consider the general form of 3×3 positive reciprocal matrices formulated in (3.1). In the consistent cases, $a = b/c$, $1/a = c/b$, $b = ac$, $1/b = 1/(ac)$, $c = b/a$, $1/c = a/b$. In the inconsistent cases, the approximation of an element by the other two elements can be considered by the grade difference

$$GD(a, b, c) = \min\Big\{\max\{|\ a - b/c\ |, |\ 1/a - c/b\ |\},$$
$$\max\{|\ b - ac\ |, |\ 1/b - 1/(ac)\ |\}, \max\{|\ c - b/a\ |, |\ 1/c - a/b\ |\}\Big\}.$$

Thus, the one grade off rule and the two grades off rule are

$$GD(a, b, c) \leq 1 \quad \text{and} \quad GD(a, b, c) \leq 2,$$

respectively.

In the case of matrices $\mathbf{A}$ of higher orders, the one grade off rule and the two grades off rule (Koczkodaj et al. 1997) are

$$GD(\mathbf{A}) = \max\Big\{GD(a, b, c) \quad \text{for each triad} \quad (a, b, c) \text{ in } \mathbf{A}\Big\} \leq 1 \text{ or } 2.$$

Figure 2 shows that the threshold $CM \leq 1/3$ corresponds to $GD \leq 2/3$, which is close to the one grade off rule.

5 Inconsistency of random pairwise comparison matrices

Golden and Wang (1990) computed the random inconsistency indices and Forman (1990) the same for incomplete *PRM*. Dodd et al. (1993) investigated the frequency distributions of random inconsistency indices and their statistical significance levels. Lane and Verdini (1989) determined the exact distribution of random inconsistency indices for 3×3 matrices, and random samples of 2,500 matrices were produced and analyzed for 4×4 to 10×10 and selected higher-order matrices, as well as stricter consistency requirements for 3×3 and 4×4 pairwise comparison matrices were suggested. Standard (2000) generated randomly 1,000 *PRM*, but restricted the CR_n as follows. For $n = 3, 4$ or 5, $CR_n < 0.1$ was required, for $n = 6$, $CR_n < 0.2$, and for $n = 7$, $CR_n < 0.3$. The computer was very slow in generating the random, low CR_n, PRM regarding sizes 6 and 7 and the results became more scattered as n increased. Additionally, regarding $n = 7$, there were no results for $CR_n < 0.1$. Due to these conditions, the low CR_n analysis was not run regarding matrices of sizes 8 and 9. A conclusion is that *Saaty*'s rule is statistically very strict for large *PRM*.

We have performed a statistical analysis of *CR* and *CM* inconsistencies. The aim of our simulation was to analyze the empirical distributions of the maximal eigenvalues $\lambda_{\max}$ of randomly generated pairwise comparison matrices. The elements a_{ij} $(i < j)$ were randomly chosen from the scale

$$\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, \frac{1}{2}, 1, 2, \ldots, 8, 9,$$

and a_{ji} is defined as $\dfrac{1}{a_{ij}}$. In the paper, the assumption of equal probabilities is used. In order to have equal probabilities $(\frac{1}{17})$, we used Matlab's *rand* function for simulating uniform distribution, the period of which is 2^{1492}. We have computed the average value of $\lambda_{\max}$ of randomly generated pairwise comparison matrices which is the basis of the mean random consistency index (RI_n). The values of $\lambda_{\max}$ corresponding to the $CR_n = 10\%$, the number of matrices which satisfies the $CR_n \leq 10\%$, $GD \leq 1$ and $GD \leq 2$ conditions were also computed. (It follows from the definition of CR_n that—if the comparisons are carried out randomly—the expected value of CR_n is 1.) In Table 4, n varies from 3 to 10, the sample size is 10^7 for all n.

In the case of 3×3 matrices, the sample size 10^7 is much larger than the number of different matrices $17^3 = 4{,}913$. Thus, many (or all) of the matrices may have been counted more than once. The ratios of the numbers of matrices holding $CR \leq 10\%$, $GD \leq 1$ and $GD \leq 2$ compared to the sample size have been also computed if each matrix counted exactly once, and found to have almost the same results as above.

Our simulations are visualized in histograms, too. Figure 3a–h show the empirical distributions of $\lambda_{\max}$ on the lower horizontal axis and the corresponding consistency ratio CR_n on the upper horizontal axis. As n increases, the shape of distribution of $\lambda_{\max}$ gets similar to a normal one in our sample. For $n = 3$, a notable part of the randomly generated matrices satisfies the $CR_n \leq 10\%$ rule. The number of matrices with $CR_n \leq 10\%$ drastically decreases as n increases (see Table 4). Regarding $n = 8, 9, 10$, we have not found a matrix in the sample of ten million with acceptable inconsistency. Based on the results, it seems that the meaning of 10% for $n = 3$ is very different from $n = 8$, which is one of the weaknesses of the inconsistency ratio by *Saaty*. It is also interesting that *consistency* and *randomness* do not exclude each other: 1.7% of 3×3 random matrices (and 0.0014% of 4×4 random matrices) are consistent.

6 Inconsistency of asymmetry

A conceptual weakness of some weighting method is related to the issue of asymmetry. The question: "To what extent does alternative i dominate j?" may be replaced by the question "To what extent is j dominated by i?" The answers to these questions are logically reciprocal. If a technique is applied first to the pairwise comparison matrix $\mathbf{A}$, yielding a solution $\mathbf{w}$, and then to the transpose $\mathbf{A}^T$, yielding a solution $\mathbf{w}'$, is $\dfrac{w_i}{w_j} = \dfrac{w'_j}{w'_i}$ for every pair (i, j)?

Eigenvector Method does not possess this asymmetry property, since the principal right and left eigenvectors of $\mathbf{A}$ are not elementwise reciprocal in the cases of inconsistent pairwise

Table 4 Average value of $\lambda_{\max}$ of randomly generated pairwise comparison matrices, RI_n, the number of matrices with $CR \leq 10\%$, $GD \leq 1$ and $GD \leq 2$

n	Sample size	Average value of $\lambda_{\max}$	RI_n	$\lambda_{\max}$ corresponding to $CR = 10\%$	Number of matrices $CR \leq 10\%$	Number of matrices $GD \leq 1$	Number of matrices $GD \leq 2$
3	10^7	4.0484	0.5242	3.1048	2.08×10^6 1.42×10^6 with $GD \leq 1$ 2.0×10^6 with $GD \leq 2$	1.42×10^6 all with $CR \leq 10\%$	2.68×10^6 2.0×10^6 with $CR \leq 10\%$
4	10^7	6.6525	0.8842	4.265	3.15×10^5 2.76×10^4 with $GD \leq 1$ 1.55×10^5 with $GD \leq 2$	2.76×10^4 all with $CR \leq 10\%$	1.7×10^5 1.55×10^5 with $CR \leq 10\%$
5	10^7	9.4347	1.1087	5.4435	2.39×10^4 61 with $GD \leq 1$ 2371 with $GD \leq 2$	61 all with $CR \leq 10\%$	2404 2371 with $CR \leq 10\%$
6	10^7	12.244	1.2488	6.6244	770 0 with $GD \leq 1$ 13 with $GD \leq 2$	0	13 all with $CR \leq 10\%$
7	10^7	15.045	1.3408	7.8045	9 0 with $GD \leq 1$ 0 with $GD \leq 2$	0	0
8	10^7	17.831	1.4004	8.9831	0	0	0
9	10^7	20.604	1.4505	10.16	0	0	0
10	10^7	23.374	1.486	11.3374	0	0	0

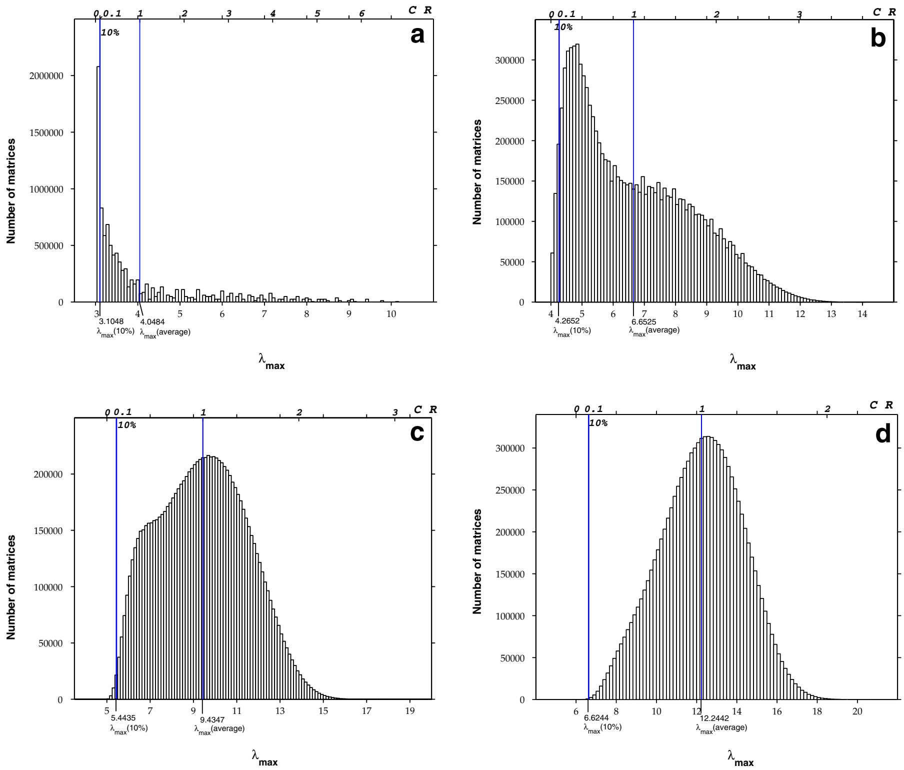


Fig. 3 $\lambda_{\max}$ and CR values of: (**a**) 3×3 random matrices, (**b**) 4×4 random matrices, (**c**) 5×5 random matrices, (**d**) 6×6 random matrices, (**e**) 7×7 random matrices, (**f**) 8×8 random matrices, (**g**) 9×9 random matrices, (**h**) 10×10 random matrices

comparison matrices. Consequently, a conceptual limitation of *EM* is the lack of asymmetry with respect to $\mathbf{A}$ and $\mathbf{A}^T$, which means that, for $n \geq 4$, there exist, generally, two competing solutions (Johnson et al. 1979). Now, it will be shown that the property of asymmetry is related to the inconsistency.

Definition 6.1 Let $\mathbf{A}$ be a pairwise comparison matrix, $\mathbf{w}$ and $\mathbf{w}'$ the priority vectors of $\mathbf{A}$ and $\mathbf{A}^T$, respectively. The invariance under transpose holds if

$$w_i \geq w_j \text{ implies } w'_i \leq w'_j, \quad \forall (i, j), \quad i, j = 1, \ldots, n. \tag{6.1}$$

It follows from the definitions that *LSM*, $\chi^2 M$, and *LLSM* defined in Table 1 always fulfil the property of invariance under transpose. *SVDM* takes this asymmetry, in some sense, into account.

Lemma 6.1 *SVDM fulfils the invariance under transpose if and only if*

$$\frac{u_i v_i + 1}{u_j v_j + 1} \geq \frac{v_i}{v_j} \textit{ implies } \frac{u_i v_i + 1}{u_j v_j + 1} \leq \frac{u_i}{u_j}, \quad \forall (i, j) \quad i, j = 1, \ldots, n, \tag{6.1}$$

where $\mathbf{u}$ *and* $\mathbf{v}$ *are the left and right singular vectors belonging to the largest singular value of* $\mathbf{A}$*, respectively.*

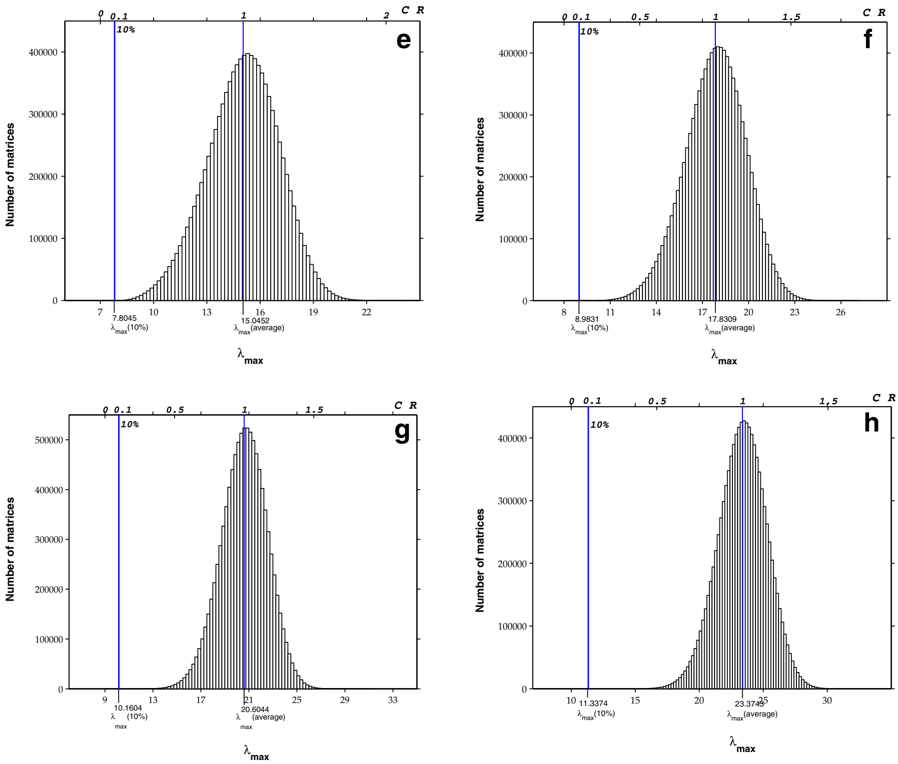


Fig. 3 continued

Proof By the formula in Table 1, the invariance under transpose holds if and only if

$$u_i + \frac{1}{v_i} \geq u_j + \frac{1}{v_j} \quad \text{implies} \quad v_i + \frac{1}{u_i} \leq v_j + \frac{1}{u_j}, \quad \forall (i, j) \quad i, j = 1, \ldots, n,$$

which is equivalent to

$$\frac{u_i v_i + 1}{u_j v_j + 1} \geq \frac{v_i}{v_j} \quad \text{implies} \quad \frac{u_i v_i + 1}{u_j v_j + 1} \leq \frac{u_i}{u_j}, \quad \forall (i, j) \quad i, j = 1, \ldots, n.$$

□

10^8 matrices of size 5×5 have been generated randomly in order to detect the rank reversals of the weights computed from the left and right eigenvectors. Based on our hypothesis, the frequency of rank reversals varies as the *CR* inconsistency ratio changes. By Table 5 and Fig. 4, the frequency of rank reversals increases as the *CR* increases. We can conclude that the larger the *CR*-inconsistency is, the more often the *EM* violates the property of invariance under transpose. Since no "cut off" point appears in Fig. 4, this seems to be another reason for reconsidering the asymmetry property.

The next example (Dodd et al. 1995) shows that a good inconsistency ratio *CR* does not exclude the rank reversal between the weights computed from the left and right eigenvectors.

Table 5 Frequency of rank reversals of the weight vectors corresponding to the left and right eigenvectors with respect to different levels of inconsistency ratio *CR*

Levels of inconsistency ratio *CR*	Number of rank reversals of the weight vectors corresponding to the left and right eigenvectors	Number of matrices	Frequency of rank reversals
$CR \leq 0.01$	8	162	0.049
$0.01 < CR \leq 0.02$	81	1138	0.071
$0.02 < CR \leq 0.03$	288	3414	0.084
$0.03 < CR \leq 0.04$	685	7130	0.096
$0.04 < CR \leq 0.05$	1253	12645	0.099
$0.05 < CR \leq 0.06$	2096	19827	0.106
$0.06 < CR \leq 0.07$	3342	29686	0.113
$0.07 < CR \leq 0.08$	5284	41400	0.128
$0.08 < CR \leq 0.09$	7896	55105	0.143
$0.09 < CR \leq 0.10$	10819	70885	0.153
$0.10 < CR \leq 0.11$	14371	88104	0.163
$0.11 < CR \leq 0.12$	18743	1.07×10^5	0.174
$0.12 < CR \leq 0.13$	23362	1.28×10^5	0.182
$0.13 < CR \leq 0.14$	27841	1.50×10^5	0.185
$0.14 < CR \leq 0.15$	33402	1.73×10^5	0.193
$0.15 < CR \leq 0.16$	39344	1.97×10^5	0.199
$0.16 < CR \leq 0.17$	44851	2.21×10^5	0.203
$0.17 < CR \leq 0.18$	50847	2.46×10^5	0.207
$0.18 < CR \leq 0.19$	57625	2.69×10^5	0.214
$0.19 < CR$	Not analyzed	9.82×10^7	Not analyzed

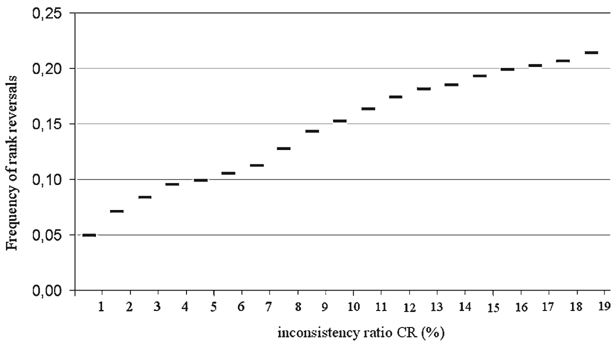


Fig. 4 Frequency of rank reversals of the weight vectors corresponding to the left and right eigenvectors with respect to different levels of inconsistency ratio *CR*

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 & 9 & 9 \\ 1 & 1 & 5 & 8 & 5 \\ 1/3 & 1/5 & 1 & 9 & 5 \\ 1/9 & 1/8 & 1/9 & 1 & 1 \\ 1/9 & 1/5 & 1/5 & 1 & 1 \end{pmatrix},$$

where $CR(\mathbf{A}) = 0.0820$, the weights of the right eigenvector

$$\mathbf{w}^T = (36.5652, 38.9564, 16.7155, 3.4693, 4.2936),$$

and the weights of the left eigenvector

$$\mathbf{w}'^T = (40.6431, 36.4208, 15.0669, 3.4391, 4.4302).$$

It is interesting that $GD(\mathbf{A}) = 4.1111$. There remain open questions, namely, how to detect and eliminate the inconsistency of asymmetry.

7 Concluding remarks

In the paper, some theoretical and numerical properties of *Saaty*'s and *Koczkodaj*'s inconsistencies of *PRM* are investigated. Based on the results, it seems that the determination of the inconsistency of *PRM* has some drawbacks, thus the improvement of the notion of inconsistency should be necessary.

Related to *Saaty*'s inconsistency ratio, some basic questions are as follows:

What is the relation between an empirical matrix from human judgements and a randomly generated one? Is an index obtained from several hundreds of randomly generated matrices the right reference point for determining the level of inconsistency of pairwise comparison matrix built up from human decisions, for a real decision problem? How to take the size of matrices into account in a more precise form?

Related to *Koczkodaj*'s consistency index, a major question seems to be the elaboration of the thresholds in higher dimensions or to replace the index by a refined grade off rule.

The existence of the inconsistency of asymmetry shows the complexity of the problem. By the example in Sect. 6, *Saaty*'s consistency of *PRM* is insufficient to exclude asymmetric inconsistency, therefore, this latter should be considered as a separate issue. Thus, it seems that only one inconsistency index is insufficient for describing the inconsistency.

Acknowledgements The authors thank the Referees for the remarks and advice. The distinction of *Saaty*'s and *Koczkodaj*'s inconsistencies, given by one of the Referees, is emphasized. This research was supported, in part, by the Hungarian Scientific Research Fund, Grant Nos. OTKA-T043276, T043241 and K60480.

References

Aguarón, J., Moreno-Jiménez, J.M.: The geometric consistency index: Approximated thresholds. Euro. J. Oper. Res. **147**, 137–145 (2003)

de Borda, J.C.: Mémoire sur les électiones au scrutin. Histoire de l'Académie Royale des Sciences, Paris (1781)

Chu, A.T.W., Kalaba, R.E., Spingarn, K.: A comparison of two methods for determining the weight belonging to fuzzy sets. J. Optim. Theory Appl. **4**, 531–538 (1979)

Condorcet, M.: Essai sur l'application de l'analyse à la probabilité des décisions rendues á la pluralité des voix, Paris (1785)

Crawford, G., Williams, C.: A note on the analysis of subjective judgment matrices. J. Math. Psychol. **29**, 387–405 (1985)

Dodd, F.J., Donegan, H.A., McMaster, T.B.M.: A statistical approach to consistency in AHP. Math. Comput. Modelling **18**, 19–22 (1993)

Dodd, F.J., Donegan, H.A., McMaster, T.B.M.: Inverse inconsistency in analytic hierarchy process. Euro. J. Oper. Res. **80**, 86–93 (1995)

Duszak, Z., Koczkodaj, W.W.: Generalization of a new definition of consistency for pairwise comparisons. Inform. Process. Lett. **52**, 273–276 (1994)

Forman, E.H.: Random indices for incomplete pairwise comparison matrices. Euro. J. Oper. Res. **48**, 153–155 (1990)

Gass, S.I., Rapcsák, T.: Singular value decomposition in AHP. Euro. J. Oper. Res. **154**(3), 573–584 (2004)

Gass, S.I., Standard, S.M.: Characteristics of positive reciprocal matrices in the analytic hierarchy process. J. Oper. Res. Soc. **53**, 1385–1389 (2002)

Golden, B.L., Wang, Q.: An alternative measure of consistency. In: Golden, B.L., Wasil, E.A., Hacker, P.T. (eds.) Analytic Hierarchy Process: Applications and Studies, pp. 68–81. Springer-Verlag (1990)

Jensen, R.E.: Comparison of eigenvector, least squares, Chi square and logarithmic least squares methods of scaling a reciprocal matrix. Trinity University, Working Paper 127 (1983). http://www.trinity.edu/rjensen/127wp/127wp.htm

Johnson, C.R., Beine, W.B., Wang, T.J.: Right-left asymmetry in an eigenvector ranking procedure. J. Math. Psychol. **19**, 61–64 (1979)

Koczkodaj, W.W.: A new definition of consistency of pairwise comparisons. Math. Comput. Modelling **8**, 79–84 (1993)

Koczkodaj, W.W., Herman, M.W., Orlowski, M.: Using consistency-driven pairwise comparisons in knowledge-based systems. In: Proceedings of the Sixth International Conference on Information and Knowledge Management, pp. 91–96. ACM Press (1997)

Lane, E.F., Verdini, W.A.: A consistency test for AHP decision makers. Decis. Sci. **20**, 575–590 (1989)

Monsuur, H.: An intrinsic consistency threshold for reciprocal matrices. Euro. J. Oper. Res. **96**, 387–391 (1996)

Murphy, C.K.: Limits on the analytic hierarchy process from its consistency index. Euro. J. Oper. Res. **65**, 138–139 (1993)

Peláez, J.I., Lamata, M.T.: A new measure of consistency for positive reciprocal matrices. Comput. Math. Appl. **46**, 1839–1845 (2003)

Saaty, T.L.: The Analytic Hierarchy Process. McGraw-Hill, New York (1980, 1990)

Saaty, T.L.: Fundamentals of Decision Making. RSW Publications (1994)

Standard, S.M.: Analysis of positive reciprocal matrices. Master's Thesis, Graduate School of the University of Maryland (2000)

Stein, W.E., Mizzi, P.J.: The harmonic consistency index for the analytic hierarchy process. Euro. J. Oper. Res. **177**, 488–497 (2007)

Thorndike, E.L.: A constant error in psychological ratings. J. Appl. Psychol. **4**, 25–29 (1920)

Thurstone, L.L.: The method of paired comparisons for social values. J. Abnormal Soc. Psychol. **21**, 384–400 (1927)

Tummala, V.M.R., Ling, H.: A note on the computation of the mean random consistency index of the analytic hierarchy process (AHP). Theory Decis. **44**, 221–230 (1998)

Vargas, L.G.: Reciprocal matrices with random coefficients. Math. Modelling **3**, 69–81 (1982)